Analysis and efficient provisioning of access networks with correlated and bursty arrivals

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SUMMARY

The statistical multiplexer problem was analyzed extensively in the discrete-time case within the context of ATM access networks or Internet. Despite the vast literature on ATM multiplexer, this problem has not been satisfactorily analyzed in the continuous-time case. In this paper, a statistical multiplexer consisting of a single-server queue is modelled and analyzed. $N$ identical Interrupted Poisson Process (IPP) arrivals flow into the multiplexer are considered, the service time is assumed to be exponentially distributed and the queue size is infinite. We developed an efficient way to approximate the superposition of $n$ IPP (or $n$ MMPP$_2$) arrival process by an MMPP$_2$ through moments and IDC (index of dispersion count) matching. We then find the closed-form Laplace transform of the departure process of an MMPP$_2$/M/1 queue. Our results prove that the departure process of a MMPP$_2$/M/1 queue is not a MMPP$_2$, which closes the long open problem that whether the departure process of an MMPP/M/1 can be a MMPP. Using Laplace transform matching, we approximated the departure process of an MMPP$_2$/M/1 by an MMPP$_2$. These results permit us to analyze a number of statistical multiplexer linked in series as they arise in an access network. Numerical experiments show that approximate analytical results match the simulation results very well. Copyright © 2013 John Wiley & Sons, Ltd.

Received …

KEY WORDS: Access networks; Markov-Modulated Poisson process; Interrupt Poisson process; Dimensioning algorithms

1. INTRODUCTION

1.1. Research Background and Related Works

The objective of this work is analysis and efficient dimensioning of an access network. Specifically, one job is to determine the size of the upstream and downstream links as a function of the number of ADSL/cable modems supported by the access network. Alternatively, given the size of the upstream and downstream links, determine how many ADSL/cable modems can be supported. The access network can be decomposed into a number of IP multiplexer. A multiplexer consists of a queue served by a single server. The queue represent the buffer at an output port where all the IP packets wait to be transmitted out, and server depicts the transmitter. The queue is fed by a large number of customers at the end-user level. What makes this problem difficult to analyze is that the traffic generated by each customer is typically bursty in nature. The multiplexer problem was analyzed in the discrete-time case during the ATM days. Despite the vast literature on ATM multiplexer, this problem was never completely analyzed. The multiplexer problem in the continuous-time case, has not been satisfactorily analyzed yet. A typical three-level hierarchy access network is shown in Figure 1. The statistical multiplexer has been widely studied in the open literature. The
A typical three-level access network

Statistical multiplexer problem was analyzed extensively in the discrete-time case within the context of ATM access networks or Internet. Despite the vast literature on ATM multiplexer, this problem has not been satisfactorily analyzed in the continuous-time case. Shah-Heydari et al. [26] used Markov-modulated Poisson processes (MMPP’s) for characterizing multimedia traffic with short-term and long-term correlation and presented a model using five parameters to approximate a superposition of N homogeneous 2-state MMPP to characterize bursty multimedia traffic. Estepa et al. [3] extended the traditional MMPP and fluid analytical models to multiplex generalized VoIP sources and proposed a simple dimensioning algorithm. Kamoun [10] considered a practical queuing system with a finite number of input links and whose arrival process is correlated, and presented performance analysis of queuing systems with correlated arrivals and service interruption. Assuming that the IPP (or MMPP₂) streams of packets from the subscribers are all identical, the resulting superposition process can be approximated by an MMPP₂, see Heffes [9] and Tian [27] and [21]. Consequently, the upstream output port of a DSLAM can be represented by an MMPP₂/M/1 queue. Now, if the interdeparture from this queue can be approximated by an MMPP₂, then the upstream output port of a Metro Ethernet switch serving a number of DSLAMs can also be modelled by an MMPP₂/M/1 queue. Finally, the upstream output port of the BRAS can be modelled in the same way. A key component in the analysis of the resulting queueing network is the characterization of the interdeparture process of an MMPP₂/M/1 queue. Once we have that, the queueing network can be easily analyzed approximately. The interdeparture process from a queue has been extensively studied. Daley [2] obtained the Laplace transform of inter-departure times of GI/M/1 queue. Heffes [9] gave the Laplace transform of inter-departure times of GI/M/N queue. Lucatoni et al.[19] showed some new results on the single server queue with a batch Markovian arrival process. Oliver et al. [20] in 1994 argued that the departure process of an MMPP/M/1 queue is not an MMPP unless the arrival process is Poisson. Bean et al. [1] pointed out there is an error in the proof given by Oliver et al.[20] and claimed that whether the departure process of an MMPP/M/1 can be a MAP (or
MMPP) is an open problem (see also reference therein). They also conjectured that the departure process of a MAP/PH/1 queue is not a MAP unless the queue is a stationary M/M/1 queue. Yeh et al. [30] introduced a recursive algorithm based on matrix-geometric solution to compute the moments of the inter-departure times of MMPP/D/1 queue with complexity at least $O(n^{2.5})$. Heindl [7] proposed a numerical approach whereby the output process of an MMPP/G/1(K) queue was approximated by a semi-Markov model which was then converted to an MMPP$_2$. This approach is computational expensive. Lim et al. [18] gave a general framework for the calculation of the Laplace transform of the inter-departure times for a single server queue with Markov renewal input and general service time distribution (MAP/G/1), which is also based on the computationally expensive matrix-geometric solution. No closed-form solution was given. Recently Tseng et al. [22] use the Markov arrival process (MAP) to model a more general input traffic that captures the correlation of interarrival times, and applying to analyze the throughput of the direct sequence-code division multiple access/unslotted ALOHA radio network with MAP. Vassilakis et al. [29] apply extended Erlang multi-rate loss model for the efficient calculation of link occupancy distribution and consequently call blocking probabilities, link utilization, and throughput per service class; the accuracy of the new model is verified by simulation. He et al. [6] propose a Markov chain model for an 802.11 LA algorithm (ONOE algorithm), aiming to identify the problems and finding the space of improvement for LA algorithms. In this paper, we analyze a statistical multiplexer consisting of a single-server queue. The service time is assumed to be exponentially distributed and the queue size is infinite. $n$ IPP arrival streams flow into the multiplexer are assumed to be identical. We first approximate the superposition of the $n$ IPP arrival streams by a two-stage MMPP (referred to as MMPP$_2$) and then we solve the resulting queue as an MMPP$_2$/M/1 queue. The analysis of the queue is done numerically. Subsequently, we develop an efficient method for approximating the departure process from the statistical multiplexer by an MMPP$_2$. Extensive numerical results have validated the accuracy of this approximation.

### 1.2. Main Contributions

The main contributions in this paper are:

- **Proposing a continuous time queueing model to analyze a number of statistical multiplexer linked in series, as they arise for instance in an ADSL access network.** Such a network consists of homes linked to a DSLAM via the telephone links. Groups of DSLAMs are linked to a metro Ethernet switch and groups of Ethernet switches are linked to a BRAS, a large router that is connected to the Internet. In this network we can recognize three statistical multiplexers. The first one represents a DSLAM that serves a number of homes, the second represents a metro Ethernet switch that serves a group of DSLAMs, and the third one represents the BRAS that serves all the metro Ethernet switches. Such a network of statistical multiplexers can be composed to individual multiplexers, which can then be analyzed in isolation.

- **Providing analysis of the characterization of the departure process from a statistical multiplexer:** In real systems, the arrival process and departure process to a queueing network can be autocorrelated. For example, the arrival process of a queueing node can be the superposed $n$ interrupted Poisson processes (IPP), which is known to be autocorrelated in general. The departure process of a single server queue with auto-correlated arrival processes, such as Markov arrival process (MAP) is not a MAP in general. In the network of queues, this causes non-product form general queueing network. Failure to take the auto-correlation in both arrival process and departure process (arrival process to the downstream queues) into consideration can lead to serious underestimation of the performance measures.

- **Providing an efficient way to approximate the superposed $n$ IPP (or 2-state Markov modulated Poisson process called MMPP$_2$) arrival process by a MMPP$_2$; and approximating the departure process by another MMPP$_2$ using the Laplace transform match technique for the first time.**

- **Finding the closed-form Laplace transform of the departure process of MMPP$_2$/M/1 queue, and showing that the departure process of a MMPP$_2$/M/1 queue is not a MMPP$_2$, which...**
closes the long open problem that whether the departure process of an MMPP/M/1 can be a MMPP.

- Exact numerical results validate that our first approximation of superposed $n$ $MMPP_2$ by another $MMPP_2$ is very accurate. Simulation results of departure process match very well analytical results of our second approximation of departure process by $MMPP_2$.

1.3. The Organization of Remaining Contents

The rest of this paper is organized as follows: Section 2 obtains the Laplace transform of the inter-arrival times of an $MMPP_2$, Section 3 describes the superposition of $n$ interrupted Poisson processes, Section 4 introduces an approximate method for the superposition of $n$ interrupted Poisson processes, Section 5 discusses the autocorrelation function of the superposed arrival process and self-similarity nature of the $MMPP_2$ process, Section 6 analyzes the $n$*IPPP/M/1 queue, Section 7 analyzes the departure process of $MMPP_2$/M/1 queue, Section 8 discusses dimensioning approach for access networks and presents typical numerical examples, then a conclusion is provided in Section 9.

2. THE LAPLACE TRANSFORM OF THE INTER-ARRIVAL TIMES OF AN $MMPP_2$

In this section we obtain the Laplace transition of the p.d.f. (probability density function) of the inter-arrival time of a two-state Markov modulated Poisson process, hereafter referred to as $MMPP_2$. Let

$$Q = \begin{bmatrix} -\delta_1 & \delta_1 \\ \delta_2 & -\delta_2 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

(1)

The $MMPP_2$ is a renewal process only in very special case (such as IPP). This may be intuitively seen as follows (see Fischer et al. [4]): Consider two consecutive arrivals in an $MMPP_2$. Suppose the state of the continuous-time Markov chain $J(t)$ is 1 at the first arrival and 2 at the second arrival. In between these arrivals there is a first transition from state 1 to state 2 via several steps, followed by a geometric number of returns to state 2 during which no arrivals occur, followed by an arrival in state 2. Clearly, each of these distributions depends on 1 and 2. Thus the time between arrivals is not exponential and the distribution of the time between the $(k-1)$st and $k$th arrivals depends on the state of $J(t)$ at the time of the $(k-1)$st and the $k$th arrivals. The Laplace transform of the p.d.f. of the inter-arrival time of an $MMPP_2$ can be obtained as follows. A transition probability matrix for the embedded Markov renewal process is defined in Fischer et al. [4]: let $J_0$ be the state of $J(t)$ at time $t=0$ and $X_0=0$. Associate with the $k$th arrival of the MMPP the corresponding state $J_k$ of the underlying Markov process as well as the time $X_k$ between the $(k-1)$st and the $k$th arrivals. Then the sequence $(J_n, X_n), n \geq 0$ is a Markov renewal sequence with transition probability matrix

$$f(x) = \int_0^x \left[ \text{exp}(Q - \Lambda)u \right] du \Lambda = \left( I - \text{exp}((Q - \Lambda)t) \right)(\Lambda - Q)^{-1} \Lambda$$

(2)

Define

$$\lambda = \lambda_1, \lambda_2, \pi = \delta_2, \delta_1/(\delta_1 + \delta_2), p = \pi \Lambda / (\pi \lambda).$$

(3)

where $\pi$ is the stationary vector of matrix $Q$ and $p$ is the stationary vector of matrix $(\Lambda - Q)^{-1} \Lambda$ which is called the transition probability matrix of the Markov chain embedded at arrival epochs (see for example Fischer et al. [4]). The Laplace transform matrix $F(s)$ of the transition probability matrix $f(x)$ is then given by

$$F(s) = E[\text{exp}(-sX)] = (s I - Q + \Lambda)^{-1} \Lambda.$$  

(4)

where $I$ is the identity matrix. The Laplace transform of the p.d.f. of the inter-arrival time of the $MMPP_2$ can be obtained as follows:

$$f_{nMMPP_2}^*(s) = pF(s) e = p(s I - Q + \Lambda)^{-1} \Lambda e.$$  

(5)
After some calculations we have:

\[
f'_{\text{MMPP}}(s) = \frac{(\delta_2 \lambda_1^2 + \delta_1 \lambda_2^2) s + \delta_2^2 \lambda_1^2 + \delta_2 \lambda_1^2 \lambda_2 + 2 \delta_1 \lambda_2 \delta_2 \lambda_1 + \delta_1^2 \lambda_2^2 + \delta_1 \lambda_2^2 \lambda_1}{(s^2 + (\lambda_1 + \lambda_2 + \delta_1 + \delta_2)s + \lambda_1 \lambda_2 + \lambda_1 \delta_2 + \lambda_2 \delta_1)(\lambda_1 \delta_2 + \delta_1 \lambda_2)}. \tag{6}
\]

or

\[
f'_{\text{MMPP}}(s) = \frac{(\delta_2 \lambda_1^2 + \delta_1 \lambda_2^2)}{s^2 + (\lambda_1 + \lambda_2 + \delta_1 + \delta_2)s + \lambda_1 \lambda_2 + \lambda_1 \delta_2 + \lambda_2 \delta_1}. \tag{7}
\]

The moments of the inter-arrival times in an MMPP can be obtained by differentiating the above Laplace transform. These moments can be used in a moment matching approach to fit an MMPP model from collected data or superposition of different arrival processes. We will show application examples later. Since the IPP is a special case of the MMPP, the Laplace transform of the inter-arrival times of an IPP with parameters ($\delta_1, \delta_2, \lambda_1$) can be obtained by setting $\lambda_2 = 0$. We have:

\[
f_{\text{IPP}}(s) = \frac{\lambda_1 (s + \delta_2)}{s^2 + (\lambda_1 + \delta_1 + \delta_2)s + \lambda_1 \delta_2}. \tag{8}
\]

To the best of our knowledge, this is perhaps the most efficient way to obtain Laplace transform of p.d.f of inter-arrival time of IPP.

3. THE SUPERPOSITION OF $n$ INTERRUPTED POISSON PROCESSES

Generally speaking, there are two ways we can study the packet stream generated by a single source: a) study the number of arrivals during a fixed time interval, and b) study the inter-arrival times between successful packet arrivals. There may be three general ways to analyze the superposition of $n$ independent identical distributed (IID) arrival processes: (1) study the number of arrivals ( for example, counts) in an interval $(0,t)$; (2) characterize the inter-arrival time distributions (such as the probability density function (p.d.f.)); (3) approximate the $n$ IPP by an MMPP. In this paper, we will combine above three methods to characterize the superposition of $n$ independent identical interrupted Poisson processes (IPP). It is well known that a single IPP is a renewal process with independent identical distributed (IID) inter-arrival times but the superposition of $n$ IPP is not a renewal process in general. In this section, we will obtain the p.d.f. of the superposed process in closed-form and then compute its moments. We will approximate this superposition with an MMPP in the following section. Whitt in [24] and Albin in [13] introduced the general formula to find the c.d.f (cumulative density function) of the inter-arrival times of the superposed process using the stationary-interval method of the superposed process. Let the stationary interval in the $i$th component process have a c.d.f $F_i$ with $j$th moment $\mu_{ij}$ and let the intensity of the $i$th counting process be $\lambda_i = \mu_{i1}$. Then the mean $\mu$ and its c.d.f $F$ satisfy

\[
\mu^{-1} = \lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n \tag{9}
\]

and

\[
1 - F(x) = \sum_{i=1}^{n} (\lambda_i / \lambda) [1 - F_i(x)] \prod_{j, j \neq i} \lambda_j \int_{x}^{\infty} [1 - F_j(s)] ds. \tag{10}
\]

Let $T$ be a random variable with the above c.d.f. Then for $k \geq 1$, its moments can be obtained as follows:

\[
E[T^k] = \int_{0}^{\infty} k x^{k-1} [1 - F(x)] dx \tag{11}
\]

For the superposition of $n$ homogeneous IPPs, if we set the c.d.f. excessive function [24]

\[
F_c(t) = \lambda_1 \int_{0}^{t} [1 - F_1(u)] du, t \geq 0, \tag{12}
\]
where $F_1(t)$ is the c.d.f. of a single arrival process which can be an arbitrary MMPP process. Applying above formulae (1)-(4), we obtain the c.d.f of the inter-arrival times in the superposed process as

$$F(t) = 1 - (1 - F_1(t))(1 - F_e(t))^{n-1}, t \geq 0,$$

(13)

and we can obtain moments of the inter-arrival times of superposed process accordingly once the c.d.f of a single IPP is given. We know that (for example, from Kuczura [14]) the p.d.f of a single IPP can be written as

$$f(t) = pe^{-\mu_1 t} + (1-p)e^{-\mu_2 t}, t \geq 0,$$

(14)

where $\mu_1$, $\mu_2$ and $p$ are obtained from $(\lambda, \delta_1, \delta_2)$, the three parameters of IPP

$$\mu_1 = 0.5(\lambda + \delta_1 + \delta_2 + \sqrt{(\lambda + \delta_1 + \delta_2)^2 - 4\lambda\delta_2})$$

(15)

$$\mu_2 = 0.5(\lambda + \delta_1 + \delta_2 - \sqrt{(\lambda + \delta_1 + \delta_2)^2 - 4\lambda\delta_2})$$

(16)

$$p = (\lambda - \mu_2)/(\mu_1 - \mu_2)$$

(17)

From the p.d.f of a single IPP, we can obtain its c.d.f and the c.d.f $F_e(t)$ readily, then we can compute the c.d.f of the inter-arrival time of superposed process using above formula. What is the computation complexity to obtain the moments of the inter-arrival times of the superposed process? Using today’s software such as Matlab and Maple, for a few hundreds of independent arrival processes, only a few seconds are needed to obtain the first three moments and variance exactly. In 1980s, researchers often used approximations to obtain these moments, see for example Whitt [24] and references therein. So the computation complexity can be considered as constant for the given parameters of individual arrival. What can we do with the p.d.f of the superposed process and its moments? We can approximate the superposed process by moment matching and we discuss this in the following section.

4. APPROXIMATION OF THE SUPERPOSITION OF $n$ IPPS BY AN MMPP$_2$

Traditionally, the arrival process to a queue is often considered as a renewal process. However, the superposed arrival process of $n$ independent IPP (or MMPP$_2$) is not a renewal process since it is correlated. Many studies are based on direct Markov chain approach for the superposition, see Fischer et al. [4]. However, this approach leads to a large number of states. For instance, the superposition of $n$ homogeneous IPPs (MMPP$_2$) is a $(n+1)$-state Markov chain and the superposition of $n$ Heterogeneous IPPs (MMPP$_2$) results to a $2^n$-state Markov chain. It is intractable to solve the problem using direct Markov chain approach for large $n$. In this section, we approximate the superposed process by an MMPP$_2$ which captures the autocorrelation of the superposed arrival processes by matching moments and the index of dispersion counts (IDC). The first moment $(m_1)$ and second moment $(m_2)$ of the inter-arrival times of an MMPP$_2$ can be obtained by differentiating the Laplace transform of the p.d.f. of the inter-arrival time distribution given by (5). We have:

$$m_1 = \frac{\delta_1 + \delta_2}{\delta_1 \lambda_1 + \delta_2 \lambda_2}$$

(18)

$$m_2 = \frac{2(\delta_1^2 + \delta_1 \lambda_1 + 2\delta_1 \delta_2 + \delta_2 \lambda_2 + \delta_2^2)}{(\delta_1 \lambda_2 + \lambda_1 \delta_2 + \lambda_1 \lambda_2)(\delta_1 \delta_2 + \lambda_2 \delta_1)}$$

(19)

However, the third moment $(m_3)$

$$m_3 = \frac{6(\delta_1^3 + 3\delta_2 \delta_1^2 + 2\delta_2^2 \lambda_1 + 2\delta_2 \delta_1 \lambda_2 + 3\delta_1 \delta_2^2 + 2\delta_2 \delta_1 \lambda_1 + \delta_1 \lambda_1^2 + \lambda_2^2 \delta_2 + 2\delta_2^2 \lambda_2 + \delta_2^3)}{(\delta_1 \lambda_2 + \lambda_1 \delta_2 + \lambda_1 \lambda_2)^2(\delta_1 \delta_2 + \lambda_1 \lambda_2)}$$

(20)

is little complicated to use for moment matching. Another parameter that can be applied is the index of dispersion count (IDC). Considering the process as a counting process, the number of arrivals in
an interval of length $t$ can be obtained from generating functions (see for example [4]). The time-dependent variable $IDC(t)$ is defined as the variance of the number of arrivals in an interval of length $t$ divided by the mean number of arrivals in $t$:

$$IDC(t) = \frac{\text{Var}(N_t)}{E(N_t)}$$  \hspace{1cm} (21)

where $N_t$ indicates the number of arrivals in an interval of length $t$. The IDC has been so defined in order that for a Poisson process the value of the IDC is 1 for all $t$. Heffes and Lucantoni [9] derive a formula for the IDC of an $\text{MMPP}_2$ process:

$$IDC(t) = 1 + \frac{2\delta_1\delta_2(\lambda_1 - \lambda_2)^2}{(\delta_1 + \delta_2)^2(\lambda_1\delta_2 + \lambda_2\delta_1)} - \frac{2\delta_1\delta_2(\lambda_1 - \lambda_2)^2(1 - \exp(-(\delta_1 + \delta_2)t))}{(\delta_1 + \delta_2)^3(\lambda_1\delta_2 + \lambda_2\delta_1)t}$$  \hspace{1cm} (22)

The asymptote of the IDC is

$$IDC_\infty = 1 + \frac{2\delta_1\delta_2(\lambda_1 - \lambda_2)^2}{(\delta_1 + \delta_2)^2(\lambda_1\delta_2 + \lambda_2\delta_1)},$$  \hspace{1cm} (23)

and it is straightforward to verify that (see Gusella et al. [5])

$$\frac{IDC_\infty - IDC_{t_0}}{IDC_\infty - 1} = 1 - \frac{\exp(-(\delta_1 + \delta_2)t_0)}{(\delta_1 + \delta_2)t_0}$$  \hspace{1cm} (24)

The quantity $r = \delta_1 + \delta_2$ can be interpreted as the “rate” at which the IDC approaches its asymptote (see Gusella et al. [5]). This equation can be used to estimate $r$ for a superposed arrival process (or measured arrival process) since the lefthand side can be easily evaluated from a point at $t_0$ on the IDC and the estimated IDC asymptote; $r$ can then be obtained by solving equation 23 numerically. Notice that the time point $t_0$ should be chosen carefully (not near to the starting point) to have a better result.

Summarizing, we have four parameters namely the first moment $m_1$, the second moment $m_2$, the index of dispersion count (IDC) and the rate $r$. So we can use these four parameters to match a $\text{MMPP}_2$ with four variables ($\delta_1, \delta_2, \lambda_1, \lambda_2$) by solving the four equations. We already discussed how to obtain the first moment, second moment and rate $r$ from the superposed arrival processes. Now we will discuss how to obtain the IDC of the superposed arrival processes. For the superposition of $n$ independent Markov arrival process (for example IPP), the IDC$(t)$ is defined

$$IDC_n(t) = \frac{\sum_{i=1}^{n} Var(N_i(t))}{\sum_{i=1}^{n} E(N_i(t))}$$  \hspace{1cm} (25)

where $E(N_i(t))$ and $Var(N_i(t))$ are the mean number of arrivals and its variance in the given interval for source $i$. If the individual arrival process is homogeneous and independent, we know that $IDC_n(t) = IDC(t)$, i.e., IDC of superposed processes will be same as the the single source. This is a nice feature for us to take advantage of when matching the superposed arrival processes to a single $\text{MMPP}_2$.

5. THE AUTOCORRELATION FUNCTION OF THE SUPERPOSED ARRIVAL PROCESSES

Let us define sequences inter-arrival times (random variables) $X_1, X_2, ..., X_n$. The term burstiness has often been used in the literature to characterize the relative variability of a given traffic when compared to that of a Poisson process. (We know that the variance divided by the mean of number of arrivals in an interval of length $t$ is 1 for Poisson process). The term correlation usually refers to the dependence that exists among successive packet inter-arrival times or to the dependence in the average packet arrival rates in successive time intervals. To measure the dependence between these random variables, the autocovariance function (or the autocorrelation
function) can be applied. The superposed arrival process of \( n \) independent arrival process in many cases (even for homogeneous IPP) will not be renewal process, i.e., the inter-arrival time will not be independent but correlated. Once we approximate the superposed arrival process as an \( \text{MMPP}_2 \), we can analyze the autocovariance quantitatively using autocovariance function \( \text{Cov}[k] \) (see [4] for example) as follow:

\[
\text{Cov}[k] = E[(X_1 - E(X_1))(X_{k+1} - E(X_{k+1}))]
\]

\[
\text{Cov}[k] = p\lambda A^{-2}\lambda (\lambda - Q)^{-1} \Lambda^{k-1} - e^p\lambda A^{-2}\Lambda = A\sigma^k
\]

where

\[
A = \frac{(\lambda_1 - \lambda_2)^2\delta_1\delta_2}{(\lambda_2\delta_1 + \lambda_1\delta_2)^2(\lambda_1\lambda_2 + \lambda_2\delta_1 + \lambda_1\delta_2)}
\]

\[
\sigma = \frac{\lambda_1\lambda_2}{\lambda_1\lambda_2 + \lambda_2\delta_1 + \lambda_1\delta_2}
\]

We can see that the correlation is null when one of the arrival rates is zero (for example IPP). The autocorrelation coefficient function of an \( \text{MMPP}_2 \) can be computed as follows:

\[
\rho[k] = \frac{E[X(t)X(t+k)] - m_1E[X(t+k)]}{\sigma^2}
\]

where \( E[X] \) is the expectation of the random variable \( X \) which can be computed using its definition equation and \((m_1, \sigma)\) is the first moment and variance of inter-arrival times of \( \text{MMPP}_2 \) respectively.

### 5.1. Self-similarity nature of the \( \text{MMPP}_2 \) process

It is known that \( \text{MMPP}_2 \) is a good model for the superposition of packet voice streams, see Lazarou et al. [15] and reference therein. The model can be used to capture the high variability of traffic over a range of small time scales. Let \( H \) be the Hurst parameter. For general self-similar processes, the Hurst parameter measures the degree of “self-similarity”. If 0.5 < \( H \) < 1, then the process has a long-range dependence (LRD), and if 0 < \( H \) ≤ 0.5, then it has a short-range dependence (SRD). \( H \) is widely used to capture the intensity of long-range dependence of a random process. The closer \( H \) is to 1, the more long-range dependent the traffic is, and vice versa [15]. Lazarou et al. [15] found a relationship between \( H \) and IDC(t) as follows:

\[
H_x(t) = \frac{d\log(\text{IDC}(t))}{dt} + \frac{1}{2}
\]

where \( H_x(t) \) is called index of variability and \( \frac{d\log(\text{IDC}(t))}{dt} \) is the local slope of the IDC curve at each \( t \) when plotted in log-log scale. Note that the index of variability is so defined in order that for a long-range dependent process \( H_x(t)=H \in (0.5, 1) \) for \( t \geq t_0 \geq 0 \). If a process is exactly self-similar, then \( H_x(t)=H \in (0.5, 1) \) for all \( t \). The index of variability can be thought of as the Hurst parameter defined at each time scale. For \( \text{MMPP}_2 \), the index of variability (or Hurst parameter) can be then obtained easily since IDC(t) is given in equation (22). \( H_x(t) \) is explicitly given in [15] as

\[
H_x(t) = 0.5\{\frac{A[1 -(1+rt)e^{-rt}]}{(1+ra)t-A(1-e^{-rt})} + 1\}
\]

where \( r = \delta_1 + \delta_2 \) and \( A = \frac{2\delta_2(\lambda_2 - \lambda_1)^2}{r^2\lambda_2} \). So that

\[
H_x(\infty) = 0.5
\]

We can then measure the self-similarity nature of \( \text{MMPP}_2 \).
6. THE DEPARTURE PROCESS OF $\text{MMPP}_2/M/1$ QUEUE

6.1. Analysis of $n^*\text{IPP}/M/1$ queue

The $n^*\text{IPP}/M/1$ queue can be analyzed numerically using matrix-geometric approach. In addition, a numerical block-matrix-power approach for the analysis of an $n^*\text{IPP}/M/m+r$ model was described in [27], which is exact and efficient for medium size $m+r$ (for example less than 200). Since it involves matrix multiplication and inversion, its complexity is at least $O(mn^3)$ which is computationally expensive for larger size problem. An alternative way to analyzing the $n^*\text{IPP}/M/1$ queue is to approximate the $n^*\text{IPP}$ by a $\text{MMPP}_2$ and then analyze numerically using the matrix-geometric approach. We provide some numerical examples of approximating $n^*\text{IPP}$ by a $\text{MMPP}_2$ to a single server queue in the numerical example section.

6.2. The departure process of $\text{MMPP}_2/M/1$ queue

In this section, we characterize the departure process of an $\text{MMPP}_2/M/1$ queue by finding the explicit Laplace transform of the inter-departure times of the $\text{MMPP}_2/M/1$ queue. To the best of our knowledge, such a characterization has not been reported in the literature. Daley [2] developed the Laplace transform of inter-departure times of GI/M/1 queue. Heffes [9] found the Laplace transform of inter-departure times of GI/M/N queue. Bean et al. [1] claimed that whether the departure process of an $\text{MMPP}/M/1$ can be a MAP (or $\text{MMPP}$) is an open problem (see reference therein). They also conjectured that the departure process of a MAP/PH/1 queue is not a MAP unless the queue is a stationary M/M/1 queue. Yeh et al. [30] introduced a recursive algorithm based on matrix-geometric solution to compute the moments of the inter-departure times of $\text{MMPP}/D/1$ queue with complexity at least $O(n^{2.5})$. Heindl [7] proposed a numerical approach whereby the output process of an $\text{MMPP}/G/1/(K)$ queue was approximated by a semi-Markov model which was then converted to an $\text{MMPP}_2$, this approach is computational expensive. Lim et al. [18] gave a general framework for the calculation of the Laplace transform of the inter-departure times for a single server queue with Markov renewal input and general service time distribution (MAP/G/1), which is also based on the computationally expensive matrix-geometric solution. No closed-form solution was given. Below, we obtain a closed-form solution of the Laplace transform of the departure process of $\text{MMPP}_2/M/1$ queue. We then equate the obtained Laplace transform to the Laplace transform of an $\text{MMPP}_2$ from where we obtain the four parameters for $\text{MMPP}_2 (\delta_1, \delta_2, \lambda_1, \lambda_2)$. We call this method the Laplace transform matching method. The $\text{MMPP}_2$ characterization of the departure process from an $\text{MMPP}_2/M/1$ queue is approximate, as in the subsequent section, we show that the output process of an $\text{MMPP}_2/M/1$ is not an $\text{MMPP}_2$ (or MAP) in general. However, through extensive numerical examples, given at the end of this paper, we show that the $\text{MMPP}_2$ characterization is a very good approximation. The Laplace transform of the probability density function of the inter-departure time from an $\text{MMPP}_2/M/1$ is of the form:

$$D(s) = (1 - \pi_0 e) H(s) + \pi_0 B(s)e$$

(34)

where $H(s)$ is the Laplace transform of service time distribution, $e$ is the unity vector, $\pi_0$ was obtained by Lucantoni [19] using the embedded Markov chain approach:

$$\pi_0 = \frac{1 - \rho}{\lambda_m} g(\Lambda - Q)$$

(35)

$\pi_0$ is defined as the probability that a departure leaves the system behind with no customer and $\rho$ is the mean offered load, $\lambda_m$ is the mean arrival rate and $g$ is defined as $gG = g$ and $ge = 1$, and for $\text{MMPP}_2$,

$$G = \left[\begin{array}{cc} 1 - G_1 & G_1 \\ G_2 & 1 - G_2 \end{array}\right] = \left[\begin{array}{cc} 1 - x & \frac{\delta_2 x}{\delta_1 + (\lambda_1 - \lambda_2)x} \\ \frac{\delta_1 x}{\delta_1 + (\lambda_1 - \lambda_2)x} & 1 - \frac{\delta_1 x}{\delta_1 + (\lambda_1 - \lambda_2)x} \end{array}\right]$$

(36)

where $x$ satisfies

$$x = 1 - \frac{\delta_2 x}{\delta_1 + (\lambda_1 - \lambda_2)x} - H(\delta_1 + \delta_2 + \lambda_1 x + \frac{\lambda_2 \delta_2 x}{\delta_1 + (\lambda_1 - \lambda_2)x})$$

(37)
For the $MMPP_2$, $G_{ij}$ is the probability that a busy period starting with the $MMPP_2$ in state $i$ ends in state $j$. For $MMPP_2$, we can find $(G_1, G_2)$ explicitly using the definition of $G$ as:

$$G = \int_0^\infty e^{(Q - \Lambda + \Lambda G)t} \, dt$$

(38)

After simplification, we obtain $G_2$ as the smallest positive root of the following equation

$$(\lambda_2 \lambda_1 - \lambda_2^2) y^3 + (\mu (\lambda_1 - \lambda_2) - \lambda_2 (\lambda_1 - \delta_1 - 2\delta_2 - \lambda_2) + \lambda_1 \delta_2) y^2 + (-\delta_2 (\delta_1 + \mu + \delta_2 + \lambda_1 - 2\lambda_2)) y + \delta_2^2$$

(39)

and $G_1$ is given by

$$G_1 = \frac{\delta_1 G_2}{\lambda_2 G_2 + \delta_2 - \lambda_1 G_2}$$

(40)

We note that there appears to be an error in the expression for $G_1$ in [4] where an iterative process for solving $G_1, G_2$ is described. Once $(G_1, G_2)$ are obtained, stationary vector $g$ can be obtained explicitly. $B(s)$ in expression (34) is defined as the Laplace transform of the idle time distribution function matrix where $B_n(t)_{ij}$ is the probability that a departure left the system empty and the arrival process in state $i$, the next departure occurs no later than time $t$ with the arrival process in state $j$, leaving $n$ customers in the system given a departure at time 0. The explicit form for $B(s)$ is obtained using the equation in Lucantoni [19]

$$B(s) = [s I - (Q - \Lambda)]^{-1} AA(s)$$

(41)

and

$$A(s) = \int_0^\infty e^{-st} e^{Qt} h(t) dt$$

(42)

where $h(t)$ is the probability density function of service time. For an $MMPP_2$, $e^{Qt}$ can be found through matrix exponential as

$$e^{Qt} = e^\pi - \frac{e^{(\delta_1 + \delta_2)t}}{\delta_1 + \delta_2} Q$$

(43)

Then $D(s)$ has a closed-form solution for the given $H(s)$ and $B(s)$. We observe that $D(s)$ is not the Laplace transform of an $MMPP_2$ but of mixture of negative exponential distributions. After simplification, $D(s)$ has the following form in general:

$$D(s) = \frac{as^2 + bs + c}{s^3 + ds^2 + es + f} = \frac{a_1}{s + \mu_1} + \frac{b_1}{s + \mu_2} + \frac{c_1}{s + \mu_3}$$

(44)

where $a, b, c, d, e, f$ is given respectively as follows:

$$a = \mu (1 - \Pi_0 e)$$

(45)

$$b = \mu (\delta_1 (1 - \Pi_0 e) + \delta_2 (1 - \Pi_0 e)) + \lambda_1 + \lambda_2 - \Pi_0 \lambda_1 - \Pi_0 \lambda_2$$

(46)

$$c = \mu (\lambda_2 \lambda_1 + \lambda_1 \delta_2 + \delta_1 \lambda_2)$$

(47)

$$d = \delta_1 + \delta_2 + \lambda_1 + \lambda_2 + \mu$$

(48)

$$e = \delta_1 \lambda_2 + \delta_2 \lambda_1 + \lambda_1 \lambda_2$$

(49)

$$f = \mu (\delta_1 \lambda_2 + \delta_2 \lambda_1 + \lambda_1 \lambda_2)$$

(50)

and $(a_1, b_1, c_1)$ can be obtained using partial fraction expansion, for example $a_1$ is the value of $(s + \mu_1) D(s)$ for $s = -\mu_1$. We note that this is the Laplace transform of a hyper-exponential distribution with three stages (Which may be converted to a three-stage Cointian distribution). Therefore, the inverse of $D(s)$, i.e., the p.d.f. of the inter-departure time is as follows:

$$d(t) = a_1 e^{-\mu_1 t} + b_1 e^{-\mu_2 t} + c_1 e^{-\mu_3 t}$$

(51)
where \((a_1, b_1, -c_1)\) and \((\mu_1, \mu_2, \mu_3)\) are all positive real numbers. We have observed numerically that the term \(c_1 \exp(-\mu_3 t)\) is very small in most cases and therefore we can approximate \(d(t)\) by the p.d.f. of another \(\text{MMPP}_2\). (If the term \(c_1 \exp(-\mu_3 t)\) is larger than 10% of the total weight, we can use moments matching to find another \(\text{MMPP}_2\) to approximate the departure process). We have:

\[
D(s) = \frac{a_1}{s + \mu_1} + \frac{b_1}{s + \mu_2}
\]

(52)

This is equivalent to the Laplace transform of a hyperexponential distribution with two terms. Comparing \(D(s)\) to the Laplace transform of the p.d.f. of an \(\text{MMPP}_2\), we obtain the following equations

\[
\frac{(\delta_2 \lambda_1^2 + \delta_1 \lambda_2^2)}{\lambda_1 \delta_2 + \delta_1 \lambda_2} = a, \lambda_1 \lambda_2 + \lambda_1 \delta_2 + \lambda_2 \delta_1 = b, \lambda_1 + \lambda_2 + \delta_1 + \delta_2 = c
\]

(53)

Together with a normalized equation for the transition rates \((\delta_1 + \delta_2)\), we can find four parameters \((\lambda_1, \lambda_2, \delta_1, \delta_2)\) for an \(\text{MMPP}_2\). We can compare the analytical results with the simulation results. Simulation results are obtained by fitting the data (the inter-departure time) to an \(\text{MMPP}_2\) using an algorithm introduced in Li et al. [16]. To compare the analytical results and simulation results, we fit both results to a single server queue and compare the mean number of customers in the system and mean response times. We find that the analytical results match the simulation results very well for many different cases. Some typical results are provided in the numerical section. Notice that the above approach also works for an \(\text{MMPP}/\text{GI}/1\) queue if the Laplace transform of the service time distribution is known.

6.3. Proof that the departure process of an \(\text{MMPP}_2/\text{M}/1\) queue is not \(\text{MMPP}_2\)

It is conjectured that the departure process of an \(\text{MMPP}/\text{M}/1\) is not \(\text{MMPP}\) (or \(\text{MAP}\)) in [20]. However it is pointed out by Bean et al. [1] that there is an error in the proof given by Olivier et al. [20]. So it is still an open problem as to whether the departure process of \(\text{MMPP}/\text{M}/1\) is not \(\text{MMPP}\) (MAP). In this section, a brief proof that the departure process of an \(\text{MMPP}_2/\text{M}/1\) is not an \(\text{MMPP}_2\) is given. Similar approach may be used for \(\text{MMPP}\) with more states.

1). Using equation for \(A(s)\) and \(B(s)\) and parameters for \(\text{MMPP}_2\), we obtain that \(b(s) = x_0 B(s) e\) which has following form:

\[
b(s) = \frac{\mu(a_1 s + b_1)}{(s + \mu)(s^2 + s \delta_2 + s \lambda_2 + s \delta_1 + \delta_1 \lambda_2 + \lambda_1 s + \lambda_1 \delta_2 + \lambda_1 \lambda_2)}
\]

(54)

2). Using equation for \(D(s)\) and value for \(\pi_0\), we can obtain \(D(s)\) which has following form in general:

\[
D(s) = \frac{\mu(a_2 s^2 + b_2 s + c_2)}{(s + \mu)(s^2 + s \delta_2 + s \lambda_2 + s \delta_1 + \delta_1 \lambda_2 + \lambda_1 s + \lambda_1 \delta_2 + \lambda_1 \lambda_2)}
\]

(55)

where

\[
a_2 = 1 - \pi_0 e
\]

(56)

and

\[
b_2 = \delta_1 (1 - \pi_0 e) + \delta_2 (1 - \pi_0 c) + \lambda_1 + \lambda_2 - \pi_{02} \lambda_1 - \pi_{01} \lambda_2
\]

(57)

\[
c_2 = \lambda_2 \lambda_1 + \lambda_1 \delta_2 + \delta_1 \lambda_2
\]

(58)

It can be simplified as

\[
D(s) = \frac{a s^2 + b s + c}{s^3 + d s^2 + e s + f}
\]

(59)

3). Comparing the Laplace transform of the inter-arrival time of an \(\text{MMPP}_2\) with \(D(s)\), we can find that \(D(s)\) is different from the Laplace transform of an \(\text{MMPP}_2\) in general. We conclude that the departure process of \(\text{MMPP}_2/\text{M}/1\) queue is not \(\text{MMPP}_2\) in general. We notice that in special case that \(\text{MMPP}_2\) has same arrival rates at two states, i.e., \(\text{MMPP}_2\) becomes a Poisson process,
we can show that the departure process is also a Poisson process with same average rate as arrival process. This is consistent with the well known result that the departure process of $M/M/1$ is a Poisson process.

7. DIMENSIONING APPROACHES AND NUMERICAL EXAMPLES

A typical dimensioning example of access network is shown in Figure 2. The main logic for performance analysis and dimensioning of the 3-level hierarchy access network is as follows:  

1) For access level, $n$ independent IPP sources are assumed, which can approximated by a single $MMPP_2$.  
2) For the 2nd-level (Ethernet switch) level, the departure process of each $MMPP_2$ from the access level is approximated by another $MMPP_2$ so we can assume that there are $k$ $MMPP_2$ sources enter into 2nd-level switch. The superposition of $k$ $MMPP_2$ sources can be approximated by a single $MMPP_2$ again.  
3) For the 3rd-level, IP router, similar approaches as the 2nd-level can be applied. Since both Ethernet switch and IP router can be modeled as a single server queue with $MMPP_2$ arrivals, the classic dimensioning approach for $MMPP_2/M/1$ can be applied. The detail dimensioning approaches are given in the following sections.

7.1. Dimensioning $MMPP_2/G/1$ queue

A closed-form solution for mean waiting time of MMPP/G/1 queue is provided in Fischer et al. [4]. We can use it to dimension the system capacity to meet the mean waiting time requirement. We introduced the dimensioning formula in the previous paper. When the $MMPP_2$ is offered to a
single server with service time Laplace transform $H(s)$ and finite first two moments, $m_1$ and $m_2$, the Laplace transform of the waiting time $W(s)$ is given in [4] by applying MMPP/G/1 queueing model. We have:

$$W(s) = s(1 - \rho)g[sI + Q - \Lambda(1 - H(s))]^{-1}e \tag{60}$$

and the mean waiting time $w$ is given in [4]

$$\frac{1}{\rho} \frac{1}{2(1 - \rho)} (2\rho + \lambda_m m_2 - 2m_1((1 - \rho)g + m_1\Pi\Lambda)(Q + \epsilon\Pi)^{-1}\lambda_e) - 0.5\lambda_m m_2], \tag{61}$$

where $e$ is a $2 \times 1$ vector of ones, $\lambda_e = (\lambda_1, \lambda_2)$, $\rho = \lambda_m/\mu$ is the offered load, and $g = [g_1, 1 - g_1]$ is a vector that can be found by an algorithm provided in [4] and also in equation (53). Using an iterative approach we can find the optimal capacity (or service rate $\mu$) so that the mean waiting time requirement is met. A fast numerical iterative procedure can then be applied to dimension the system to meet the mean delay requirement $W_{IPP}$. If the mean and variance (or second moment) of the inter-arrival times and the service times are known, then an upper bound of the mean waiting time in the queue is given by the expression (using result from GI/G/1 queue [12])

$$W_{IPP} \leq \frac{\lambda_a(\sigma^2_A + \sigma^2_S)}{2(1 - \rho)} \tag{62}$$

where $\sigma^2_A$ and $\sigma^2_S$ are the variance of interarrival time and service time distribution respectively, $\lambda_a$ is the mean arrival rate, $\rho$ is the offered load which is product of mean arrival rate and mean service time. It is known that this upper bound will work well when the offered load is close to 1 (heavy traffic approximation). We may also dimension the system to meet the mean delay time requirement by using this upper bound iteratively.

In the following, we provide some numerical examples to validate our proposed approaches on approximating the superposition of $n$ IPP arrivals and approximating the departure process of the $MMPP_2/M/1$ process. We observed that the queueing performance such as the mean number in the system and response time can be very different for IPP and $MMPP_2$ arrival process, even if IPP has the same moments of the inter-arrival time as the $MMPP_2$. This is because the IPP and $MMPP_2$ have different auto-variance functions (The IPP has an auto-variance equal to zero) which will cause the queueing performance to be different. 

**Example 1:** In this example, we consider an $n * IPP/M/1$ queue with $n = 10$ and service rate $\mu$ is changing so that the offered load of the single server queue is changing from 0.1 to 0.9. The single IPP has parameters $(r_1, r_2, \lambda) = (0.9, 0.1, 1.0)$ which makes it very bursty. We analyze this queue exactly using matrix-geometric procedure. We also approximate the superposition of $n * IPP$ by an $MMPP_2$ using our proposed approach and subsequently analyze this queue as an $MMPP_2/M/1$ queue using the matrix-geometric solution. The $MMPP_2$ has parameters $(r_1, r_2, \lambda_1, \lambda_2) = (0.2712, 0.7288, 9.5786, 7.4446)$. The error is the relative error in percentage computed by (exact-approximation)/exact \times 100 \% . The results are given in Table 1. We observe that the exact results and approximation results match very well. 

**Example 2:** This is the same example as above, only $n = 20$. The results are given in Table 2. The equivalent $MMPP_2$ has parameters $(r_1, r_2, \lambda_1, \lambda_2) = (0.3271, 0.6729, 18.9355, 16.0759)$. 

**Example 3:** same as in example 1, with $n = 100$. The equivalent $MMPP_2$ has parameters $(r_1, r_2, \lambda_1, \lambda_2) = (0.4180, 0.5820, 92.5426, 86.4603)$. The results are given in Table 3. We observed that:

1. The auto-covariance is (0.0236, 0.0166, 0.6852e-003) for $n = (10, 20, 100)$ respectively. That is, the auto-covariance is decreasing when $n$ is increasing. This is consistent with the known results that superposition of infinite (or very large) number of IPPs will be a Poisson process which has auto-covariance equal to zero. We actually can find a $n$ so that the superposition of $n$ IPP will be close to a Poisson process.

2. The queueing performance (the mean waiting time and number in the system) is decreasing when $n$ is increasing and other parameters are the same. This means the auto-covariance affects/dominates the queueing performance.
Table I. Comparison of the mean waiting time $W$ and the mean number of customers $N$ in the system obtained by exact and $MMPP_2$ approximation for $n=10$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$W$ (Exact)</th>
<th>$W$ ($MMPP_2$)</th>
<th>Error(%)</th>
<th>$N$(Exact)</th>
<th>$N$(MMPP$_2$)</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0230</td>
<td>0.0225</td>
<td>2.17</td>
<td>0.1230</td>
<td>0.1225</td>
<td>0.41</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1101</td>
<td>0.1063</td>
<td>3.45</td>
<td>0.3101</td>
<td>0.3063</td>
<td>1.23</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2896</td>
<td>0.2805</td>
<td>3.14</td>
<td>0.5896</td>
<td>0.5805</td>
<td>1.54</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5975</td>
<td>0.5841</td>
<td>2.24</td>
<td>0.9975</td>
<td>0.9841</td>
<td>1.34</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0976</td>
<td>1.0819</td>
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</tr>
<tr>
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<td>7.2222</td>
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<td>0.9</td>
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<td>0.09</td>
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<td>16.7108</td>
<td>0.08</td>
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</table>

Table II. Comparison of the mean waiting time $W$ and the mean number $N$ of customers in the system obtained by exact and $MMPP_2$ approximation for $n=20$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$W$ (Exact)</th>
<th>$W$ ($MMPP_2$)</th>
<th>Error(%)</th>
<th>$N$(Exact)</th>
<th>$N$(MMPP$_2$)</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0086</td>
<td>0.0085</td>
<td>0.26</td>
<td>0.1172</td>
<td>0.1169</td>
<td>1.16</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0416</td>
<td>0.0403</td>
<td>0.99</td>
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<td>0.2805</td>
<td>3.12</td>
</tr>
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<td>0.3</td>
<td>0.1143</td>
<td>0.1093</td>
<td>1.89</td>
<td>0.5286</td>
<td>0.5186</td>
<td>4.37</td>
</tr>
<tr>
<td>0.4</td>
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<td>2.37</td>
<td>0.8971</td>
<td>0.8758</td>
<td>4.27</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4794</td>
<td>0.4630</td>
<td>2.25</td>
<td>1.4588</td>
<td>1.4260</td>
<td>3.42</td>
</tr>
<tr>
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<td>0.8538</td>
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<td>2.3491</td>
<td>2.3077</td>
<td>2.37</td>
</tr>
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<td>0.28</td>
<td>16.4736</td>
<td>16.4274</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Table III. Comparison of the mean waiting time $W$ and mean number $N$ of customers in the system obtained by exact and $MMPP_2$ approximation for $n=100$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$W$ (Exact)</th>
<th>$W$ ($MMPP_2$)</th>
<th>Error(%)</th>
<th>$N$(Exact)</th>
<th>$N$(MMPP$_2$)</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0012</td>
<td>0.0012</td>
<td>0.00</td>
<td>0.1123</td>
<td>0.1123</td>
<td>0.00</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0057</td>
<td>0.0057</td>
<td>0.00</td>
<td>0.257</td>
<td>0.2567</td>
<td>0.12</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0152</td>
<td>0.015</td>
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<td>0.452</td>
<td>0.4502</td>
<td>0.40</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0331</td>
<td>0.0324</td>
<td>2.11</td>
<td>0.7313</td>
<td>0.724</td>
<td>1.00</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0663</td>
<td>0.064</td>
<td>3.47</td>
<td>1.1627</td>
<td>1.1403</td>
<td>1.93</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1288</td>
<td>0.1235</td>
<td>4.11</td>
<td>1.8875</td>
<td>1.8345</td>
<td>2.81</td>
</tr>
<tr>
<td>0.7</td>
<td>0.2533</td>
<td>0.2436</td>
<td>3.83</td>
<td>3.2331</td>
<td>3.1364</td>
<td>2.99</td>
</tr>
<tr>
<td>0.8</td>
<td>0.5362</td>
<td>0.5226</td>
<td>2.54</td>
<td>6.1616</td>
<td>6.026</td>
<td>2.20</td>
</tr>
<tr>
<td>0.9</td>
<td>1.4533</td>
<td>1.4386</td>
<td>1.01</td>
<td>15.4328</td>
<td>15.2855</td>
<td>0.95</td>
</tr>
</tbody>
</table>

3). For the same $n$, when the offered load is increasing, the mean waiting time and number in the system are both increasing; this is consistent with theoretical results. Example 4: In this example, we show the results of the superposition of $n$ homogeneous $MMPP_2$. We use the single $MMPP_2$ with parameters $(r_1, r_2, \lambda_1, \lambda_2) = (0.9, 0.1, 10, 1)$. We obtain the infinitesimal generator $Q$ and $\Lambda$. Using matrix-geometric approach, we found exact solution for the $n \times MMPP_2/M/1$ queue for $n=20$. Through moments and IDC,
Table IV. Comparison of exact and approximate results of the superposition of 20 homogeneous \( MMPP_2 \)

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>N(Exact)</th>
<th>N(Appr)</th>
<th>Error(%)</th>
<th>W(Exact)</th>
<th>W(Appr)</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1125</td>
<td>0.1125</td>
<td>0.00</td>
<td>0.0003</td>
<td>0.0003</td>
<td>0.00</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2584</td>
<td>0.2584</td>
<td>0.00</td>
<td>0.0015</td>
<td>0.0015</td>
<td>0.00</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4585</td>
<td>0.4584</td>
<td>0.02</td>
<td>0.0042</td>
<td>0.0042</td>
<td>0.00</td>
</tr>
<tr>
<td>0.4</td>
<td>0.7574</td>
<td>0.7574</td>
<td>0.54</td>
<td>0.0095</td>
<td>0.0094</td>
<td>1.05</td>
</tr>
<tr>
<td>0.5</td>
<td>1.2993</td>
<td>1.2736</td>
<td>1.98</td>
<td>0.0203</td>
<td>0.021</td>
<td>3.33</td>
</tr>
<tr>
<td>0.6</td>
<td>2.385</td>
<td>2.4284</td>
<td>1.79</td>
<td>0.0481</td>
<td>0.0469</td>
<td>2.49</td>
</tr>
<tr>
<td>0.7</td>
<td>5.0443</td>
<td>5.2608</td>
<td>4.29</td>
<td>0.1143</td>
<td>0.12</td>
<td>4.99</td>
</tr>
<tr>
<td>0.8</td>
<td>11.6971</td>
<td>12.6715</td>
<td>8.33</td>
<td>0.2868</td>
<td>0.3122</td>
<td>8.86</td>
</tr>
<tr>
<td>0.9</td>
<td>34.5901</td>
<td>36.5382</td>
<td>5.63</td>
<td>0.8866</td>
<td>0.9373</td>
<td>5.72</td>
</tr>
</tbody>
</table>

Table V. Comparison of analytical and simulation results of the departure process of \( MMPP_2/M/1 \)

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>Output(Ana)</th>
<th>Output(Sim)</th>
</tr>
</thead>
<tbody>
<tr>
<td>([\delta_1, \delta_1, \lambda_1, \lambda_2])</td>
<td>([\delta_1, \delta_1, \lambda_1, \lambda_2])</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>([0.9425, 0.0575, 4.9307, 0.9873])</td>
<td>([0.9219, 0.0781, 4.9013, 1.0167])</td>
</tr>
<tr>
<td>0.3</td>
<td>([0.8797, 0.1203, 3.0128, 0.9386])</td>
<td>([0.8649, 0.1351, 3.0059, 0.9456])</td>
</tr>
<tr>
<td>0.4</td>
<td>([0.7442, 0.2558, 2.0996, 0.8684])</td>
<td>([0.7305, 0.2696, 2.0993, 0.8687])</td>
</tr>
<tr>
<td>0.5</td>
<td>([0.5084, 0.4916, 1.6267, 0.7513])</td>
<td>([0.4979, 0.5022, 1.6288, 0.7492])</td>
</tr>
<tr>
<td>0.6</td>
<td>([0.2664, 0.7336, 1.3960, 0.5887])</td>
<td>([0.2612, 0.7388, 1.3983, 0.5864])</td>
</tr>
<tr>
<td>0.7</td>
<td>([0.8745, 0.1255, 0.4169, 1.2869])</td>
<td>([0.8790, 0.1210, 0.4145, 1.2893])</td>
</tr>
<tr>
<td>0.8</td>
<td>([0.9456, 0.0544, 0.2614, 1.2316])</td>
<td>([0.9474, 0.0526, 0.2599, 1.2331])</td>
</tr>
<tr>
<td>0.9</td>
<td>([0.9806, 0.0194, 0.1292, 1.2000])</td>
<td>([0.9814, 0.0186, 0.1281, 1.2011])</td>
</tr>
</tbody>
</table>

we obtained an \( MMPP_2 \) with parameters \((0.6966, 0.3034, 56.3226, 30.0507)\). Using the \( MMPP_2 \) as the input to an \( MMPP_2/M/1 \) queue, we compute the mean waiting time and mean number of customers in the system. The exact solution (which is obtained by simulating two tandem \( MMPP_2/M/1 \) queues) and the approximate results are shown in Table 4. We observe that the approximate \( MMPP_2 \) slightly overestimates the mean number and mean waiting times in the system in many cases. Example 5: In this example, we study the accuracy of the analytical results and simulation results for the departure process of an \( MMPP_2/M/1 \) queue. We do this as follows. We consider a tandem queueing network with two nodes. A very bursty \( MMPP_2 \) arrival process with parameters \((\delta_1=0.98, \delta_2=0.02, \lambda_1=10, \lambda_2=1.0)\) is considered as the input to the first node (single server with infinite buffer size and FIFO). We change the service rate \( \mu \) so that the offered load is varying from 0.2 to 0.9. The departure process from the first node is fed into the second node which is a single server infinite capacity queue, with exponentially distributed service time with a rate of \( \mu \). Our validation includes two parts. First, we compare the departure process from the first node as characterized by an \( MMPP_2 \) using our proposed approach against simulation results obtained due to in Li et al. [16] and reference therein. Secondly, we compare the mean number of customers and the mean waiting time in the second node obtained by simulation and by solving numerically the second node as an \( MMPP_2/M/1 \) queue, where \( MMPP_2 \) is the departure process from the first node as characterized above. The results are presented below. O(Sim) and O(Ana) with parameters \((\delta_1, \delta_2, \lambda_1, \lambda_2)\) are the departure processes from simulation and analytical results respectively. N(Sim) and N(Ana) are the mean number of customers in the system from simulation and analytical results respectively, and T(Sim), T(Ana) are mean response times from simulation and analytical results respectively. Example 6: We also compute the autocovariance function (ACF) for the departure process of an \( MMPP_2/M/1 \) queue. The analytical and
Table VI. Comparison of ACFs from analytical and simulation results of the departure process of an MMPP\(_2/M/1\) (2)

<table>
<thead>
<tr>
<th>(\rho)</th>
<th>(ACF_1) (Ana)</th>
<th>(ACF_1) (Sim)</th>
<th>(ACF_3) (Ana)</th>
<th>(ACF_3) (Sim)</th>
<th>IDC (Ana)</th>
<th>IDC (Sim)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.0752</td>
<td>0.0782</td>
<td>0.0482</td>
<td>0.0489</td>
<td>2.3880</td>
<td>2.6458</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0566</td>
<td>0.0569</td>
<td>0.0280</td>
<td>0.0278</td>
<td>1.7665</td>
<td>1.8104</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0416</td>
<td>0.0413</td>
<td>0.0153</td>
<td>0.0150</td>
<td>1.4877</td>
<td>1.4968</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0290</td>
<td>0.0286</td>
<td>0.0075</td>
<td>0.0073</td>
<td>1.3241</td>
<td>1.3248</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0187</td>
<td>0.0184</td>
<td>0.0032</td>
<td>0.0031</td>
<td>1.2157</td>
<td>1.2145</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0109</td>
<td>0.0105</td>
<td>0.0011</td>
<td>0.0010</td>
<td>1.1411</td>
<td>1.1375</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0050</td>
<td>0.0048</td>
<td>2.2902e-004</td>
<td>2.1833e-004</td>
<td>1.0821</td>
<td>1.0799</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0014</td>
<td>0.0013</td>
<td>1.8425e-005</td>
<td>1.7290e-005</td>
<td>1.0356</td>
<td>1.0369</td>
</tr>
</tbody>
</table>

Table VII. Comparison of exact and approximate (appr) results of the departure process of an MMPP\(_2/M/1\) for low burstiness \(H=0.52\)

<table>
<thead>
<tr>
<th>(\rho)</th>
<th>(m_1) (exact)</th>
<th>(m_2) (exact)</th>
<th>(m_3) (exact)</th>
<th>(m_1) (appr)</th>
<th>(m_2) (appr)</th>
<th>(m_3) (appr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.8475</td>
<td>1.6174</td>
<td>1.2521</td>
<td>0.8238</td>
<td>1.5453</td>
<td>1.2770</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8475</td>
<td>1.5955</td>
<td>1.2215</td>
<td>0.8416</td>
<td>1.5773</td>
<td>1.2268</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8475</td>
<td>1.5730</td>
<td>1.1902</td>
<td>0.8451</td>
<td>1.5654</td>
<td>1.1920</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8475</td>
<td>1.5503</td>
<td>1.1587</td>
<td>0.8463</td>
<td>1.5465</td>
<td>1.1593</td>
</tr>
<tr>
<td>0.6</td>
<td>0.8475</td>
<td>1.5276</td>
<td>1.1270</td>
<td>0.8468</td>
<td>1.5252</td>
<td>1.1272</td>
</tr>
<tr>
<td>0.7</td>
<td>0.8475</td>
<td>1.5048</td>
<td>1.0953</td>
<td>0.8491</td>
<td>1.5119</td>
<td>1.0969</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8475</td>
<td>1.4820</td>
<td>1.0635</td>
<td>0.8483</td>
<td>1.4855</td>
<td>1.0645</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8471</td>
<td>1.4578</td>
<td>1.0316</td>
<td>0.8480</td>
<td>1.4617</td>
<td>1.0326</td>
</tr>
</tbody>
</table>

Simulation results are shown in the Table 6. Auto-covariance function (ACF) for an MMPP\(_2\) is given in Fischer et al. [4] as

\[
ACF_{\text{MMPP}_2}(k) = p(\Lambda - Q)^{-2}\Lambda((\Lambda - Q)^{-1}\Lambda)^{k-1} - ep(\Lambda - Q)^{-2}\Lambda; \tag{63}
\]

And the IDC\((\text{IDC}_\infty)\) for an \(\text{MMPP}_2\) is given in equation (63). Example 7: Comparison of exact and approximate (appr) results of the departure process of an MMPP\(_2/M/1\) for low burstiness \(H=0.52\). Example 8: Comparison of exact and approximate (appr) results of the departure process of an MMPP\(_2/M/1\) for low burstiness \(H=0.73\). For low and medium burstiness \(H=0.51\) and \(H=0.73\) (input \(\text{MMPP}_2\) with parameters (0.5, 0.5, 1,2) and (0.9,0.1, 10,1) respectively), approximation results match exact results very well with maximum absolute error less than a few percent points. The results are shown in Table 7 and 8. Example 9: Comparison of exact and approximate (appr) results of the departure process of an MMPP\(_2/M/1\) for low burstiness \(H=0.91\). For high burstiness \(H=0.91\) with input \(\text{MMPP}_2\) (0.09, 0.01, 10,1), the approximation results match the exact results very well with maximum relative error less than 2 percent. The results are shown in Table 9. If we use the squared coefficient of variation \(C^2\) (equals to the variance divided by the squared mean) as the indicator of burstness, we obtain the results for the output processes of an \(\text{MMPP}_2/M/1\) as follows. Example 10: Comparison of exact and approximate results of the departure process of an \(\text{MMPP}_2/M/1\) for high burstness \(C^2=1.1429\). Example 11: Comparison of exact and approximate results of the departure process of an \(\text{MMPP}_2/M/1\) for high burstness \(C^2=15.6341\). Example 12: Comparison of exact and approximate results of the departure process of an \(\text{MMPP}_2/M/1\) for high burstness \(C^2=23.1331\). From Table 10-12, we observed that the exact moments and approximate moments match very well, max relative errors in all cases are less than 9 percent. Example 13: Considering two network nodes in a link, the first node is modelled as \(\text{MMPP}_2/M/1\) queue, its
departure process is fed into the second node. We compare analytical and simulation results fed the departure process of the first node to the second node with offered load 0.2. The service rate of the second node in Table 13 is set to 11.7926 so that the offered load is 0.2 for the second node (We also have results for the offered load varying from 0.1 to 0.9 for the second node). We observed that similar results from both analytical results and simulation results (we omit results when offered load is 0.3 to 0.8). Example 14: Another example is the second node with the offered load 0.9. Results
Table XI. Comparison of exact and approximate results of the departure process of an \( \text{MMPP}_2/\text{M}/1 \) for high burstness \( C^2 = 15.6341 \), \( \text{MMPP}_2(0.1, 6.1, 0.1, 0.1) \)

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( m_1 ) (exact)</th>
<th>( m_2 ) (exact)</th>
<th>( m_3 ) (exact)</th>
<th>( m_1 ) (appr)</th>
<th>( m_2 ) (appr)</th>
<th>( m_3 ) (appr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.3226</td>
<td>1.7297</td>
<td>25.4266</td>
<td>0.3226</td>
<td>1.7297</td>
<td>25.4266</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3226</td>
<td>1.7221</td>
<td>25.3080</td>
<td>0.3226</td>
<td>1.7222</td>
<td>25.3080</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3226</td>
<td>1.7098</td>
<td>25.1100</td>
<td>0.3226</td>
<td>1.7104</td>
<td>25.1100</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3226</td>
<td>1.6732</td>
<td>24.5224</td>
<td>0.3226</td>
<td>1.6793</td>
<td>24.5248</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3226</td>
<td>1.7346</td>
<td>25.5020</td>
<td>0.3226</td>
<td>1.5687</td>
<td>22.8340</td>
</tr>
<tr>
<td>0.6</td>
<td>0.3226</td>
<td>1.3573</td>
<td>19.3963</td>
<td>0.3226</td>
<td>1.3686</td>
<td>19.4018</td>
</tr>
<tr>
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<td>0.3226</td>
<td>1.0920</td>
<td>15.0360</td>
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<td>1.0946</td>
<td>15.0370</td>
</tr>
<tr>
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<td>0.8002</td>
<td>10.1860</td>
<td>0.3226</td>
<td>0.8002</td>
<td>10.1860</td>
</tr>
<tr>
<td>0.9</td>
<td>0.3226</td>
<td>0.5118</td>
<td>5.3429</td>
<td>0.3226</td>
<td>0.5118</td>
<td>5.3429</td>
</tr>
</tbody>
</table>

Table XII. Comparison of exact and approximate (appr) results of the departure process of an \( \text{MMPP}_2/\text{M}/1 \) for high burstness \( C^2 = 23.1331 \), \( \text{MMPP}_2(0.1, 6.1, 0.1, 0.1) \)

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( m_1 ) (exact)</th>
<th>( m_2 ) (exact)</th>
<th>( m_3 ) (exact)</th>
<th>( m_1 ) (appr)</th>
<th>( m_2 ) (appr)</th>
<th>( m_3 ) (appr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.5263</td>
<td>0.9499</td>
<td>2.7942</td>
<td>0.5266</td>
<td>0.9499</td>
<td>2.7943</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5263</td>
<td>0.9504</td>
<td>2.7958</td>
<td>0.5772</td>
<td>0.9491</td>
<td>2.7581</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5263</td>
<td>0.8947</td>
<td>2.5875</td>
<td>0.5274</td>
<td>0.8949</td>
<td>2.5876</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5263</td>
<td>0.8465</td>
<td>2.3907</td>
<td>0.5266</td>
<td>0.8465</td>
<td>2.3907</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5263</td>
<td>0.7979</td>
<td>2.1770</td>
<td>0.5264</td>
<td>0.7979</td>
<td>2.1771</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5263</td>
<td>0.7492</td>
<td>1.9476</td>
<td>0.5264</td>
<td>0.7492</td>
<td>1.9477</td>
</tr>
<tr>
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<td>0.5263</td>
<td>0.7004</td>
<td>1.7027</td>
<td>0.5263</td>
<td>0.7004</td>
<td>1.7027</td>
</tr>
<tr>
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<td>0.5263</td>
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<td>1.4421</td>
<td>0.5263</td>
<td>0.6516</td>
<td>1.4421</td>
</tr>
<tr>
<td>0.9</td>
<td>0.5263</td>
<td>0.6028</td>
<td>1.1662</td>
<td>0.5263</td>
<td>0.6028</td>
<td>1.1662</td>
</tr>
</tbody>
</table>

Table XIII. Comparison of analytical and simulation results fed the departure process of the first node to the second node with offered load 0.2

<table>
<thead>
<tr>
<th>( \rho ) (1st node)</th>
<th>N(Ana)</th>
<th>N(sim)</th>
<th>Error(%)</th>
<th>T (ana)</th>
<th>T (sim)</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.3065</td>
<td>0.3100</td>
<td>1.13</td>
<td>0.2349</td>
<td>0.2525</td>
<td>6.97</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2742</td>
<td>0.2746</td>
<td>0.15</td>
<td>0.2307</td>
<td>0.2244</td>
<td>2.81</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2635</td>
<td>0.2634</td>
<td>0.04</td>
<td>0.2226</td>
<td>0.2195</td>
<td>1.41</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2583</td>
<td>0.2582</td>
<td>0.04</td>
<td>0.2186</td>
<td>0.2168</td>
<td>0.83</td>
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<tr>
<td>0.6</td>
<td>0.2552</td>
<td>0.2552</td>
<td>0.00</td>
<td>0.2161</td>
<td>0.2151</td>
<td>0.46</td>
</tr>
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<td>0.2533</td>
<td>0.2532</td>
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<td>0.8</td>
<td>0.2519</td>
<td>0.2518</td>
<td>0.04</td>
<td>0.2137</td>
<td>0.2131</td>
<td>0.28</td>
</tr>
<tr>
<td>0.9</td>
<td>0.2508</td>
<td>0.2508</td>
<td>0.00</td>
<td>0.2127</td>
<td>0.2123</td>
<td>0.19</td>
</tr>
</tbody>
</table>

are shown in Table 14. We observed that the auto-covariance is decreasing from the first node when the offered load is increasing, that is why the mean number of customers in the system and mean response time of the second queue is increasing even its offered load is decreasing. That is, the queueing performance of the queue depends on the auto-covariance as well as the offered load.
Table XIV. Comparison of analytical and simulation results fed the departure process of the first node to the second node with offered load 0.9

<table>
<thead>
<tr>
<th>( \rho (1^{st} \text{node}) )</th>
<th>N(Ana)</th>
<th>N (sim)</th>
<th>Error(%)</th>
<th>T (ana)</th>
<th>T (sim)</th>
<th>Error(%)</th>
</tr>
</thead>
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<tr>
<td>0.2</td>
<td>15.0754</td>
<td>16.1806</td>
<td>6.83</td>
<td>12.4189</td>
<td>12.2580</td>
<td>1.31</td>
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<tr>
<td>0.3</td>
<td>12.3049</td>
<td>12.4876</td>
<td>1.46</td>
<td>10.3561</td>
<td>10.2039</td>
<td>1.49</td>
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<tr>
<td>0.4</td>
<td>11.0707</td>
<td>11.1063</td>
<td>0.32</td>
<td>9.3556</td>
<td>9.2521</td>
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<tr>
<td>0.5</td>
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<td>10.3579</td>
<td>4.05</td>
<td>8.7643</td>
<td>8.6976</td>
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<tr>
<td>0.6</td>
<td>9.8908</td>
<td>9.8853</td>
<td>0.06</td>
<td>8.3752</td>
<td>8.3333</td>
<td>0.50</td>
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<tr>
<td>0.7</td>
<td>9.5764</td>
<td>9.5613</td>
<td>0.16</td>
<td>8.1315</td>
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<tr>
<td>0.8</td>
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<td>9.3229</td>
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<td>9.1427</td>
<td>0.06</td>
<td>7.7576</td>
<td>7.7406</td>
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</table>

8. CONCLUSIONS

In this paper, we developed an efficient way to approximate the superposition of \( n \) IPP (or \( n \) \( \text{MMP}P_2 \)) arrival process by an \( \text{MMP}P_2 \) through moments and IDC matching. We then find the closed-form Laplace transform of the departure process of an \( \text{MMP}P_2/\text{M}/1 \) queue. Using Laplace transform matching, we approximated the departure process of an \( \text{MMP}P_2/\text{M}/1 \) by an \( \text{MMP}P_2 \). The approximate analytical results match the simulation results very well. It will be an interesting topic to extend the current work to the \( \text{MMP}P_2/\text{M}/N \) queue and other related queueing models.

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REFERENCES